Kac-Moody quantum superalgebras and global crystal bases

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Quantum groups

Let $\mathfrak{g}$ be a Kac-Moody algebra with quantum group $U_q(\mathfrak{g})$

$U_q(\mathfrak{n}^-) \subset U_q(\mathfrak{g})$ is related to Hall algebras, quantized shuffles.

$U_q(\mathfrak{g})$ has a symmetry $^-$: $q \mapsto q^{-1}$.

CANONICAL BASES

The algebra $U_q(\mathfrak{n}^-)$ has an extraordinarily nice basis.

It is

- suitably independent of choice,
- bar-invariant,
- well-behaved on the playground of integrable modules,
- "almost-orthogonal" (and can be characterized by this),
- categorifiable (cf. [Rouquier, Khovanov-Lauda]),
- just generally awesome.

For all these reasons (and more!), it deserves the honorific

The Canonical Basis

Finding the canonical bases

These bases were discovered through the work of Lusztig and Kashiwara.

Lusztig: perverse sheaves.

Kashiwara: the crystal basis at "$q = 0$".

QUANTUM GROUPS FOR SUPER

Is there a super version of this picture?

It isn't clear what geometry could be used.

There are various crystal structures in modules:

- $\mathfrak{osp}(1|2n)$ [Musson-Zou]
- $\mathfrak{gl}(m|n)$ [Benkart-Kang-Kashiwara], [Kwon]
- $\mathfrak{q}(n)$ [Grantcharov-Jung-Kang-Kashiwara-Kim]
- for KM superalgebra with "even" weights [Jeong]

Until recently, there was doubt of existence of canonical bases.

INSPIRATION FROM CATEGORIFICATION

[Wang, Ellis-Lauda-Khovanov], [Kang-Kashiwara-Tsuchioka] provide a fertile setting for categorification.

[W, EKL]: spin (nil)Hecke algebras

[KKT]: Hecke quiver superalgebras

These categorify certain quantum half KM (super)algebras and integrable modules. [Hill-Wang, Kang-Kashiwara-Oh]

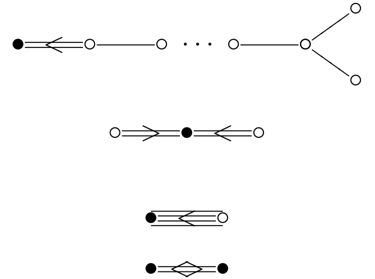
This is strong evidence for a canonical basis for KM super.

WHICH KAC-MOODY SUPERALGEBRAS?

We consider a KM superalgebra with GCM A indexed by $I = I_0 \coprod I_1$ (simple roots) and satisfying:

- $a_{ij} \in \mathbb{Z}$, $a_{ii} = 2$, $a_{ij} \leq 0$
- there exist positive symmetrizing coefficients d_i ($d_i a_{ij} = d_j a_{ji}$)
- (non-isotropy) $a_{ij} \in 2\mathbb{Z}$ for $i \in I_1$
- (bar-compatibility) $d_i \equiv_2 p(i)$

Examples



$(\mathfrak{osp}(1|2n))$

π-QUANTUM INTEGERS

There is a bar involution on $\mathbb{Q}(q)$ given by

$$q \mapsto \pi q^{-1} \quad (\pi^2 = 1)$$

- $\pi = 1 \rightsquigarrow$ non-super case.
- $\pi = -1 \rightsquigarrow$ super case.

We have bar-invariant quantum integers:

$$[n] = \frac{(\pi q)^n - q^{-n}}{\pi q - q^{-1}}, \quad [n]^!, \quad \begin{bmatrix} n \\ a \end{bmatrix} \in \mathbb{Z}[q, q^{-1}].$$

These allow us to define quantum divided powers:

$$*^{(n)} = \frac{*^n}{[n]^!}.$$

THE RANK 1 CASE

Let U be the $\mathbb{Q}(q)$-algebra generated by $E, F, K^{\pm 1}$ such that

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad EF - \pi FE = \frac{K - K^{-1}}{\pi q - q^{-1}}.$$

The bar involution is given by

$$\overline{E} = E, \quad \overline{F} = F, \quad \overline{K} = K^{-1}, \quad \bar{q} = \pi q^{-1}.$$

- $\pi = 1 \rightsquigarrow U_q(\mathfrak{sl}(2))$
- $\pi = -1 \rightsquigarrow$ "quantum $\mathfrak{osp}(1|2)$"

COMPARING SUPER VS NON-SUPER

There are many nice similarities that can be deduced without choosing π.

- U has a triangular decomposition $U = \langle E\rangle \otimes \left\langle K^{\pm 1}\right\rangle \otimes \langle F\rangle$.
- U is a Hopf (super)algebra.
- U has a quasi-R-matrix and Casimir-type element.
- U has semi-simple finite-dimensional modules.

The commutation formulas are even almost the same:

$$E^{(m)}F^{(n)} = \sum_i \pi^{mn-\binom{i+1}{2}} F^{(m-i)} \begin{bmatrix} K; 2i-n-m \\ i \end{bmatrix} E^{(n-i)}$$

COMPARING SUPER VS NON-SUPER

The largest difference is in their categories of f.d. modules.

Facts:

- ($\pi = 1$) there is a simple $U_q(\mathfrak{sl}(2))$-module of dimension n for each $n \geq 0$.
- ($\pi = -1$) there is a simple "quantum $\mathfrak{osp}(1|2)$"-module of dimension n for each odd $n \geq 0$.

This causes headaches for crystals.

ADJUSTING THE DEFINITION

Fact (Zou): "quantum $\mathfrak{osp}(1|2)$" has simple modules of all even dimensions if we extend the field to $\mathbb{Q}(\sqrt{\pi}, q)$.

Q: How can we account for all modules over $\mathbb{Q}(q)$?
A: Modify the definition of U to get an algebra U' where

$$EF - \pi FE = \frac{\pi K - K^{-1}}{\pi q - q^{-1}}$$

When $\pi = -1$, U' has only even-dimensional simples/ $\mathbb{Q}(q)$!

GLUING

Theorem (C-Wang)

For $\pi = -1$, *the algebra* $\mathcal{U} = U \oplus U'$

- *is a Hopf (super)algebra;*
- *has finite-dimensional simple modules of each dimension;*
- *has a semisimple finite dimensional representation theory;*

This has a trivial canonical basis for U^-.

But does the modified quantum group have a canonical basis?

Modified quantum algebra

Throw in idempotents 1_n to obtain a non-unital algebra.

$$(1 \rightsquigarrow \sum_n 1_n)$$

The algebra $\dot{\mathcal{U}}$ is generated by $1_n, E1_n, F1_n$ such that

$$1_m 1_m = \delta_{nm} 1_n, \quad 1_{n+2}E = E1_n, \quad 1_{n-2}F = F1_{n-2}$$

$$(EF - \pi FE)1_n = [n]1_n$$

Theorem (C-Wang)

The algebra $\dot{\mathcal{U}}$ has a canonical basis

$$\left\{ E^{(a)}1_{-n}F^{(b)}, \quad \pi^{ab}F^{(b)}1_nE^{(a)} : n \geq a+b \right\}$$

Missing link

$$
\begin{array}{ccccc}
U & \twoheadleftarrow & \color{red}{(?)} & \twoheadrightarrow & U' \\
\downarrow & & \downarrow & & \downarrow \\
\dot{\mathcal{U}} \sum 1_{2n} & \hookrightarrow & \dot{\mathcal{U}} & \hookleftarrow & \dot{\mathcal{U}} \sum 1_{2n+1}
\end{array}
$$

Expanding the Cartan

The difference between U and U' is

$$EF - \pi FE = \frac{\pi^p K - K^{-1}}{\pi q - q^{-1}}$$

where p is the parity of the "allowed weights".

If "$K = q^h$", by analogy we define "$J = \pi^h$".

Adding these elements, we obtain the definition in [C-Hill-Wang]:

Definition [CHW1]

Let $\mathfrak{g}$ be a KM superalgebra, A its symmetrizable GCM.
Let $U_q(\mathfrak{g})$ be the $\mathbb{Q}(q)$-algebra with generators E_i, F_i, $K_i^{\pm 1}$, J_i such that

$$J_i^2 = 1, \quad J_i K_i = K_i J_i, \quad J_i J_j = J_j J_i, \quad K_i K_j = K_j K_i,$$

$$J_i E_j J_i^{-1} = \pi^{a_{ij}} E_j, \quad K_i E_j K_i^{-1} = q^{a_{ij}} E_j,$$

$$J_i F_j J_i^{-1} = \pi^{-a_{ij}} F_j, \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j,$$

$$E_i F_j - \pi^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{J_i^{d_i} K_i^{d_i} - K_i^{-d_i}}{(\pi q)^{d_i} - q^{-d_i}};$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \pi^{p(k;i,j)} E_i^{(1-a_{ij}-k)} E_j E_i^{(k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k \pi^{p(k;i,j)} F_i^{(1-a_{ij}-k)} F_j F_i^{(k)} = 0,$$

where $p(k;i,j) = kp(i)p(j) + \frac{1}{2}k(k-1)p(i)$.

THE BAR INVOLUTION AND COPRODUCT

We extend $q \mapsto \pi q^{-1}$ to $U_q(\mathfrak{g})$ by setting

$$\overline{E_i} = E_i, \quad \overline{F_i} = F_i, \quad \overline{K_i} = J_i K_i^{-1}, \quad \overline{J_i} = J_i.$$

We can also define a (super) coproduct Δ by

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes K_i^{-d_i} + J_i^{d_i} \otimes E_i \\
\Delta(F_i) &= F_i \otimes 1 + K_i^{d_i} \otimes F_i \\
\Delta(K_i) &= K_i \otimes K_i \\
\Delta(J_i) &= J_i \otimes J_i
\end{aligned}$$

REPRESENTATIONS

Let P (P^+) be the set of (dominant) weights of $\mathfrak{g}$.

A weight module is a $U_q(\mathfrak{g})$-module $M = \bigoplus_{\lambda \in P} M^\lambda$, where

$$M^\lambda = \left\{ m \in M : K_i m = q^{\langle h_i, \lambda \rangle} m, \quad J_i m = \pi^{\langle h_i, \lambda \rangle} m \right\}.$$

We can define highest-weight and integrable modules as usual to obtain a category $\mathcal{O}_{\text{int}}$.

Simple modules: $V(\lambda)$ for all $\lambda \in P^+$

PROPERTIES

Proposition [CHW1]. For $\pi = \pm 1$,

- $U_q(\mathfrak{g}) = U^+ \otimes U^0 \otimes U^-$.
- $U_q(\mathfrak{g})$ is a Hopf (super)algebra.
- There is a quasi-R-matrix and quantum Casimir element.
- Each $M \in \mathcal{O}_{\text{int}}$ is completely reducible.

The question remains: is there a canonical basis for U^-?

Kashiwara operators

There is a left derivation operator e_i' and a bilinear form $(-,-)$ on U^- such that:

$$(1,1) = 1, \quad (F_i x, y) = (x, e_i'(y)).$$

Each $u \in U^-$ can be written $x = \sum F_i^{(n)} x_n$ such that $e_i'(x_n) = 0$.

We can define Kashiwara operators

$$\tilde{f}_i x = \sum F_i^{(n+1)} x_n, \quad \tilde{e}_i x = \sum F_i^{(n-1)} x_n.$$

CRYSTAL FOR U^-

Let $A \subset \mathbb{Q}(q)$ be the ring of functions with no pole at 0.

U^- is said to have a **crystal basis** (L, B) if
L is a A-lattice of U^- closed under $\tilde{e}_i, \tilde{f}_i$
and $B \subset L/qL$ satisfies

- B is a signed basis of L/qL;
- $\tilde{e}_i B \subseteq B \cup \{0\}$ and $\tilde{f}_i B \subseteq B$;
- For $b \in B$, if $\tilde{e}_i b \neq 0$ then $b = \tilde{f}_i \tilde{e}_i b$.

As in the $\pi = 1$ case, the crystal lattice/basis is unique and

$$V(\lambda) \supset L(\lambda) = \sum A\tilde{f}_{i_1} \ldots \tilde{f}_{i_n} v_\lambda, \quad B(\lambda) = \left\{\tilde{f}_{i_1} \ldots \tilde{f}_{i_n} v_\lambda + qL\right\}$$

$$(\lambda \in P^+ \cup \{\infty\}, V(\infty) = U^-)$$

WHY MUST THE BASIS BE SIGNED?

- $I = I_1 = \{i, j\}$ such that $a_{ij} = a_{ji} = 0$.

$$\tilde{f}_i \tilde{f}_j v_\lambda = \pi \tilde{f}_j \tilde{f}_i v_\lambda$$

  Should $\tilde{f}_i \tilde{f}_j v_\lambda$ or $\tilde{f}_j \tilde{f}_i v_\lambda$ be the basis element?

- **Tensor Product Rule**:

$$\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \phi_i(b) > \epsilon_i(b'), \\ \pi^{p(i)p(b)} b \otimes \tilde{f}_i b' & \text{otherwise.} \end{cases}$$

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \phi_i(b) \geq \epsilon_i(b'), \\ \pi^{p(i)p(b)} b \otimes \tilde{e}_i b' & \text{otherwise.} \end{cases}$$

EXAMPLE

$\mathfrak{g} = \mathfrak{osp}(1|2)$

$\Delta(F^{(k)}) = \sum_{r+s=k} (\pi q)^{-rs} F^{(r)} K^s \otimes F^{(s)}$

Looking at $V(n) \otimes V(1)$, (by convention, $p(v_\lambda) = 0$)

$$F^{(k)}(v_n \otimes v_1) = F^{(k)} v_n \otimes v_1 + \pi^{1-k} q^{n+1-k} F^{(k-1)} v_n \otimes F v_1.$$

Then $\tilde{f}^{(n+1)}(v_n \otimes v_1) = \pi^n (\tilde{f}^n v_n) \otimes (\tilde{f} v_1) \bmod q$.

Remark on the tensor product rule

Signs in the tensor product rule is not a new idea ([BKK]).

In earlier work, versions without the sign appear.

- In [MZ], the signs are absorbed into factors of $\sqrt{-1}$.
- In [Jeong], some signs were forgotten.

SIGNED ORTHONORMALITY

As usual, there is a polarization bilinear form $(-,-)$ which descends to a bilinear form $(-,-)_0$ on L/qL.

When $\pi = -1$, $B(\lambda)$ is a signed orthonomal basis; that is,

$$(b, b)_0 = 1 \text{ or } \pi.$$

(when $\pi = 1$, the basis is honestly orthonormal.)

Example: $\mathfrak{g} = \mathfrak{osp}(1|4)$
with simple roots α_1 (short, odd) and α_2 (long, even):

$$x = \tilde{f}_1^4 \tilde{f}_2 \cdot 1, \quad y = \tilde{f}_1^3 \tilde{f}_2 \tilde{f}_1 \cdot 1$$

$$(x, x) \in 1 + q^2 A, \quad (y, y) \in \pi + q^2 A, \quad (x, y) \in q^2 A.$$

SIGNED ORTHONORMALITY

Proposition.

- [Kashiwara] For $\pi = 1$,

$$L(\infty) = \left\{ x \in U^- | (x,x) \in A \right\}.$$

- [CHW2] For $\pi = -1$,

$$L(\infty) \neq \left\{ x \in U^- | (x,x) \in A \right\}.$$

Example Continued:
Set $z = \tilde{f}_1^4 \tilde{f}_2 \cdot 1 + \tilde{f}_1^3 \tilde{f}_2 \tilde{f}_1 \cdot 1 \in L(\infty)$.

Then $(z,z) \in q^2 A$, so $(q^{-1}z, q^{-1}z) \in A$ despite $q^{-1}z \notin L(\infty)$.

GLOBALIZING THE CRYSTAL BASIS

We want to find a map

$$G : L/qL \to U_{\mathbb{Z}}^{-} \cap L \cap \overline{L}$$

such that $G(B)$ is a global (= canonical!) basis for U^{-}.
Essential to the argument in [K] is that L is invariant under

$$\sigma : U^{-} \to U^{-} \text{ with } \sigma(xy) = \sigma(y)\sigma(x), \quad \sigma(F_i) = F_i.$$

Since $(-,-)$ is σ-invariant, the $\pi = 1$ case follows from

$$L = \left\{ x \in U^{-} | (x,x) \in A \right\}.$$

But when $\pi = -1$,

$$L \neq \left\{ x \in U^{-} | (x,x) \in A \right\},$$

so how to prove invariance?

Two Curious Solutions

There are two ways to work around this issue, each interesting in its own way.

Categorification: We can realize the crystal in a categorification.

Two-Parameter: We can directly connect the crystal bases at $\pi = \pm 1$ by passing through a two-parameter version as developed by [Fan-Li].

END OF THE TALE

Theorem (C-Hill-Wang)

The half quantum group U^- and its integrable modules admit canonical bases.

Related Papers

[HW] *Categorication of quantum Kac-Moody superalgebras*, arXiv:1202.2769, to appear in Trans. AMS.
[CW] *Canonical basis for quantum* $\mathfrak{osp}(1|2)$, arXiv:1204.3940, Lett. Math. Phys. 103 (2013), 207–231.
[CHW1] *Quantum supergroups I. Foundations*, arXiv:1301.1665.
[CHW2] *Quantum supergroups II. Canonical Basis*, arXiv:1304.7837.

Slides available at
http://people.virginia.edu/~sic5ag/

Thank you for your attention!