Do Super Cats Make Odd Knots?

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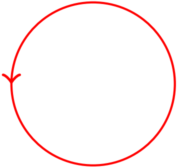
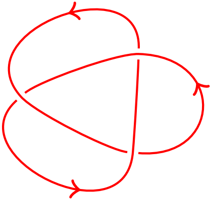
MPIM Oberseminar
November 5, 2015

WHAT IS A KNOT?



(The unknot)



(The Trefoil Knot)

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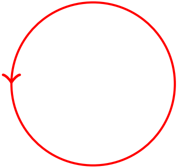

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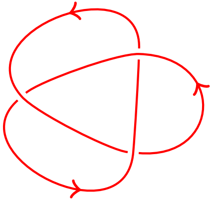
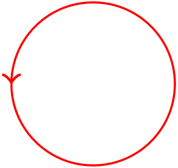


(The Trefoil Knot)

- Knots = $\{S^1 \hookrightarrow \mathbb{R}^3\}$ /isotopy
- 2D projection (avoiding triple intersections)

What is a knot?



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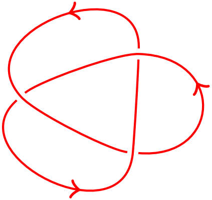


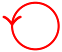
(The Trefoil Knot)

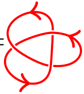
- Knots = $\{S^1 \hookrightarrow \mathbb{R}^3\}$ /isotopy
- 2D projection (avoiding triple intersections)
- Knots are isotopic iff projections equivalent under planar isotopy + Reidemeister moves
- Useful tool for distinguishing knots: invariants!

Jones Polynomial and Khovanov Homology

Example (V. Jones, 1984)

Given a knot (or *link*) diagram D, there is a Laurent polynomial $J_D = J_D(q)$ that is an invariant of knots.

$D =$  has $J_D = q + q^{-1}$.

$D =$  has $J_D = -q^{-9} - q^{-7} + q^{-5} + 2q^{-3} + q^{-1}$.

Thus the trefoil is not the unknot!

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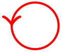
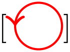
$D =$  has $J_D = q + q^{-1}$.

Example (Khovanov, 2000)

For a knot diagram D, construct complex $[D]$ of graded v.s./k, subject to rules similar to Jones polynomial:

[] $= 0 \to \underbrace{k[1] \oplus k[\text{-}1]}_{\text{hdeg}=0} \to 0$ "=" $q + q^{-1}$

Khovanov Homology (KH) is the homology of this complex.
The graded Euler characteristic of KH = Jones polynomial!

REPRESENTATION THEORY

Example (Reshetikhin-Turaev, late 1980's)

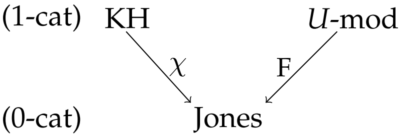
Knots can be encoded in a category TAN of *tangles*.
Given a "nice" Hopf algebra H and module V, can find a functor from TAN to H-REP. This defines a operator invariant of the knot.

Special Case:

The quantum group $U_q(\mathfrak{sl}_2)$ is a "nice enough" Hopf algebra.
This procedure with simple 2-dim module yields a map $\mathbb{Q}(q) \to \mathbb{Q}(q)$.
Evaluation at 1 is the Jones polynomial!

CATEGORIFICATION

Both examples are *categorifications*:



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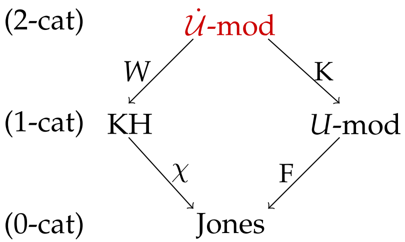

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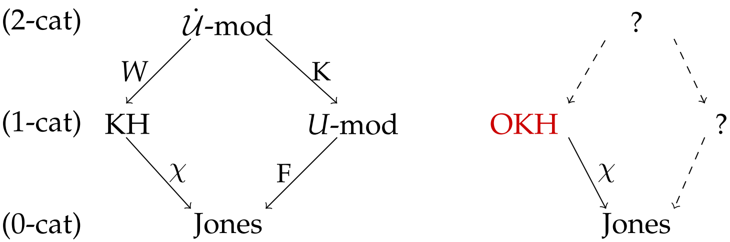

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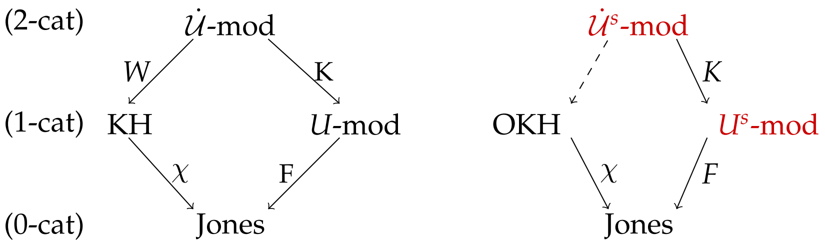

- Linked via *categorified quantum groups* (for all colored invariants)
- Question: Can we find similar "explanation" for OKH?
- Conjecture: Yes, with quantum $\mathfrak{osp}(1|2n)$ (Lie *super*algebra)

What is U^s

Let $U^s = U_q(\mathfrak{osp}(1|2)) = \mathbb{Q}(q)\left\langle E, F, K, K^{-1}, J \right\rangle$ with rel'ns

$$KK^{-1} = 1,\ KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,\ EF + FE = \frac{JK - K^{-1}}{-q - q^{-1}}$$

$$J^2 = 1 \text{ and } J \text{ is central.}$$

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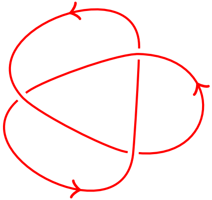
There are important module homomorphisms:

1. $R : X \otimes Y \cong Y \otimes X$ ($\mathcal{R}$ matrix) for any X, Y; satisfies braid rel'ns.
2. There is a simple 2-dim. module V.
   $\mathbb{Q}(q) \xrightarrow{\epsilon} V^* \otimes V \xrightarrow{\delta} \mathbb{Q}(q), \quad \mathbb{Q}(q) \xrightarrow{\epsilon'} V \otimes V^* \xrightarrow{\delta'} \mathbb{Q}(q)$

$$\delta \circ \epsilon = q + \pi q^{-1} = \pi \delta' \circ \epsilon'$$

KNOT DIAGRAMS TO MORPHISMS

Translate a knot diagram $D \leftrightsquigarrow$ a map $\mathbb{Q}(q) \to \mathbb{Q}(q)$ (constant):



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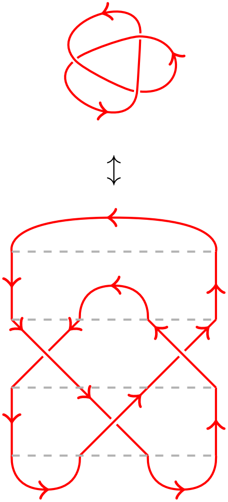

- Cut diagram into simple pieces

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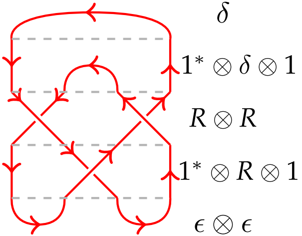

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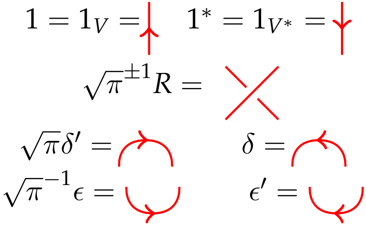

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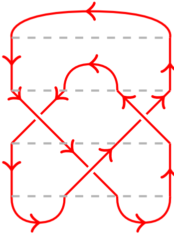


$$\delta \circ 1^* \otimes \delta \otimes 1 \circ R \otimes R \circ 1^* \otimes R \otimes 1 \circ \epsilon \otimes \epsilon$$

- Cut diagram into simple pieces
- Translate each slice into a morphism

$1 = 1_V =$  $1^* = 1_{V^*} =$ 

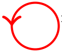
$\sqrt{\pi}^{\pm 1} R =$ 

$\sqrt{\pi}\delta' =$  $\delta =$ 

$\sqrt{\pi}^{-1}\epsilon =$  $\epsilon' =$ 

- Compose and scale by $(\pi q)^{\text{writhe}}$

Then we get the Jones polynomial in the variable $\sqrt{\pi}^{-1}q$!

Example:  $= \sqrt{\pi}^{-1}(q + \pi q^{-1}) = \sqrt{\pi}^{-1} q + (\sqrt{\pi}^{-1} q)^{-1} =$ 

Higher rank and/or colored invariants

Theorem (C)

Let K be a knot, $V(\lambda)$ a f.d. irrep. of $U^{\mathrm{ns}} = U_q(\mathfrak{so}(1+2n))$ or $U^{\mathrm{s}} = U_q(\mathfrak{osp}(1|2n))$, and $J_K^{\mathrm{s/ns}}(q)$ the corresponding colored knot invariant. Then $J_K^{\mathrm{s}}(q) = \sqrt{-1}^{*} J_K^{\mathrm{ns}}(\sqrt{-1}q)$.

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Main idea in proof:

- $\exists$ Complex isomorphism $\psi : \widehat{\mathcal{U}}^{\mathrm{ns}} \cong \widehat{\mathcal{U}}^{\mathrm{s}}$ with $\psi(q) = \sqrt{-1}q$.
- ψ induces a nice functor Ψ on a rep category
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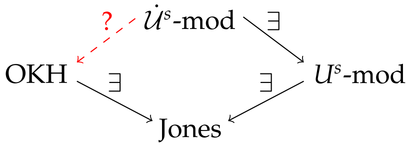
Conclusion: $\mathcal{U}^{\mathrm{s}}$ does not give new invariants. ☹

But it may lead to new odd knot homologies! ☺

Current Interests

- Construct an odd analogue of Webster's construction. (An answer for the Jones polynomial would be nice!)



- Studying these quantum groups at roots of unity.
- Further study of other types of quantum superalgebras.
- Categorification of quantum superalgebras and reps.

Thanks for your attention!

Selected References:

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P. Ozsváth, J. Rasmussen, and Z. Szabó, *Odd Khovanov homology*, arXiv:0710.4300

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