Quantum supergroups and canonical bases

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$U_q(\mathfrak{g})$ is the $\mathbb{Q}(q)$ algebra with generators E_i, F_i, $K_i^{\pm 1}$ for $i \in I$,
Various relations; for example,

- $K_i \approx q^{h_i}$, e.g. $K_i E_j K_i^{-1} = q^{\langle h_i, \alpha_j \rangle} E_j$
- quantum Serre, e.g. $F_i^2 F_j - [2] F_i F_j F_i + F_j F_i^2 = 0$
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Some important features are:

- an involution $\overline{q} = q^{-1}, \overline{K}_i = K_i^{-1}, \overline{E}_i = E_i, \overline{F}_i = F_i$;
- a bar invariant integral $\mathbb{Z}[q, q^{-1}]$-form of $U_q(\mathfrak{g})$.

Canonical basis and categorification

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Relation to **categorification**:

- $U_q(\mathfrak{n}^-)$ categorified by quiver Hecke algebras [Khovanov-Lauda, Rouquier]
- canonical basis $\leftrightarrow$ indecomp. projectives (symmetric type) [Varagnolo-Vasserot].

Lie superalgebras

$\mathfrak{g}$: a Lie superalgebra (everything is $\mathbb{Z}/2\mathbb{Z}$-graded).
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Example: $\mathfrak{osp}(1|2)$ is the set of 3×3 matrices of the form

$$A = \underbrace{\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & c & d \\ 0 & e & -c \end{array}\right)}_{A_0} + \underbrace{\left(\begin{array}{c|cc} 0 & a & b \\ \hline -b & 0 & 0 \\ a & 0 & 0 \end{array}\right)}_{A_1}$$

with the super bracket; i.e. the usual bracket, except
$[A_1, B_1] = A_1 B_1 + B_1 A_1$.

(Note: The subalgebra of the A_0 is $\cong$ to $\mathfrak{sl}(2)$.)

Our question

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Is there a canonical basis à la Lusztig, Kashiwara?

Some potential obstructions are:

- Existence of isotropic simple roots: $(\alpha_i, \alpha_i) = 0$
- No integral form, bar involution (e.g. quantum $\mathfrak{osp}(1|2)$)
- Lack of positivity due to super signs

Experts did not expect canonical bases to exist!

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- [KKO12]: QHSA's categorify quantum groups (Generalizes a rank 1 construction of [EKL11])
- [HW12]: QHSA's categorify quantum *super*groups (assuming no isotropic roots)

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Key Insight [HW]: use a parameter $\pi^2 = 1$ for super signs
e.g. a super commutator $AB + BA$ becomes $AB - \pi BA$

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There is a bar involution on $\mathbb{Q}(q)[\pi]$ given by $q \mapsto \pi q^{-1}$.

$$[n] = \frac{(\pi q)^n - q^{-n}}{\pi q - q^{-1}}, \text{ e.g. } [2] = \pi q + q^{-1}.$$

Note $\pi q + q^{-1}$ has positive coefficients. (vs. $-q + q^{-1}$)

(Important for categorification: e.g. $F_i^2 = (\pi q + q^{-1})F_i^{(2)}$.)

Anisotropic KM

$I = I_0 \coprod I_1$ (simple roots), parity $p(i)$ with $i \in I_{p(i)}$.

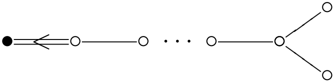
Symmetrizable generalized Cartan matrix $(a_{ij})_{i,j \in I}$:

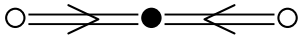
- $a_{ij} \in \mathbb{Z}$, $a_{ii} = 2$, $a_{ij} \leq 0$;
- positive symmetrizing coefficients d_i ($d_i a_{ij} = d_j a_{ji}$);
- (anisotropy) $a_{ij} \in 2\mathbb{Z}$ for $i \in I_1$;
- (bar-compatibility) $d_i = p(i)$ mod 2, where $i \in I_{p(i)}$

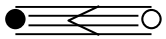
Examples (Finite and Affine)

(●=odd root)

$(\mathfrak{osp}(1|2n))$







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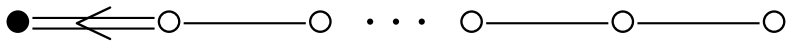

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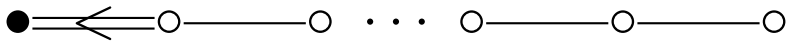

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These algebras have similar representation theories.

- $\mathfrak{osp}(1|2n)$ irreps $\leftrightarrow$ half of $\mathfrak{so}(2n+1)$ irreps.
- $U_q(\mathfrak{osp}(1|2n))/\mathbb{C}(q) \leftrightarrow$ all of $U_q(\mathfrak{so}(2n+1))$ irreps. [Zou98]

Rank 1

[CW]: $U_q(\mathfrak{osp}(1|2))/\mathbb{Q}(q)$ can be tweaked to get all reps.

$$EF - \pi FE = \underbrace{\frac{1K - K^{-1}}{\pi q - q^{-1}}}_{\text{even h.w.}} \quad \text{or} \quad \underbrace{\frac{\pi K - K^{-1}}{\pi q - q^{-1}}}_{\text{odd h.w.}}$$

Rank 1

[CW]: $U_q(\mathfrak{osp}(1|2))/\mathbb{Q}(q)$ can be tweaked to get all reps.

$$EF - \pi FE = \frac{\pi^h K - K^{-1}}{\pi q - q^{-1}} \quad (h \text{ the Cartan generator}) \qquad (*)$$

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New definition: generators $E, F, K^{\pm 1}, J$, relations

$$J^2 = 1, \quad JK = KJ,$$

$$JEJ^{-1} = E, \quad KEK^{-1} = q^2 E, \quad JFJ^{-1} = F, \quad KFK^{-1} = q^{-2} F,$$

$$EF - \pi F_j E_i = \frac{JK - K^{-1}}{\pi q - q^{-1}}; \qquad (*')$$

(If h is the Cartan element, $K = q^h$ and $J = \pi^h$.)

Definition of quantum covering groups

Let A be a symmetrizable GCM. U is the $\mathbb{Q}(q)[\pi]$-algebra with generators E_i, F_i, $K_i^{\pm 1}$, J_i and relations

$$J_i^2 = 1, \quad J_iK_i = K_iJ_i, \quad J_iJ_j = J_jJ_i$$

$$J_iE_jJ_i^{-1} = \pi^{a_{ij}}E_j, J_iF_jJ_i^{-1} = \pi^{-a_{ij}}F_j.$$

$$E_iF_j - \pi^{p(i)p(j)}F_jE_i = \delta_{ij}\frac{J_i^{d_i}K_i^{d_i} - K_i^{-d_i}}{(\pi q)^{d_i} - q^{-d_i}};$$

and others (super quantum Serre, usual K relations).

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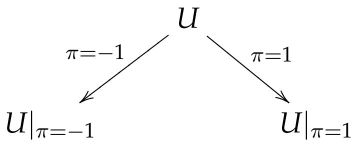
and others (super quantum Serre, usual K relations).

Bar involution: $\overline{q} = \pi q^{-1}, \overline{K_i} = J_i K_i^{-1}, \overline{E_i} = E_i, \overline{F_i} = F_i$

Can also define a bar-invariant integral $\mathbb{Z}[q, q^{-1}, \pi]$-form!

RELATION TO QUANTUM (SUPER)GROUPS

By specifying a value of π, we have maps



- $U|_{\pi=1}$ is a quantum group (forgets $\mathbb{Z}/2\mathbb{Z}$ grading).
- $U|_{\pi=-1}$ is a quantum supergroup.

Representations

X: integral weights, X^+: dominant integral weights.

A weight module is a U-module $M = \bigoplus_{\lambda \in X} M_\lambda$, where

$$M_\lambda = \left\{ m \in M : K_i m = q^{\langle h_i, \lambda \rangle} m, \quad J_i m = \pi^{\langle h_i, \lambda \rangle} m \right\}.$$

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Example: $U_q(\mathfrak{osp}(1|2))$, $X = \mathbb{Z}$, $X^+ = \mathbb{N}$ and $M = \bigoplus_{n \in \mathbb{Z}} M_n$.

$$Jm = \pi^n m, \quad Km = q^n m \quad (m \in M_n)$$

REPRESENTATIONS

Can define highest-weight (h.w.) and integrable (int.) modules.

Theorem (C-Hill-Wang)

For each $\lambda \in X^+$, there is a unique simple ("π-free") module $V(\lambda)$ of highest weight λ. Any ("π-free") h.wt. int. M is a direct sum of these $V(\lambda)$.

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Example: $U_q(\mathfrak{osp}(1|2))$ has simple π-free modules $V(n)$, which are free $\mathbb{Q}(q)[\pi]$-modules of rank $n+1$. (Like $\mathfrak{sl}(2)$!)

$$V(n) = \underbrace{V(n)|_{\pi=1}}_{\dim_{\mathbb{Q}(q)}=n+1} \oplus \underbrace{V(n)|_{\pi=-1}}_{\dim_{\mathbb{Q}(q)}=n+1}$$

Approaches to canonical bases

Two potential approaches to constructing a canonical basis:

- [Lusztig] using geometry
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There are various crystal structures in modules:

- $\mathfrak{osp}(1|2n)$ [Musson-Zou] ('98)
- $\mathfrak{gl}(m|n)$ [Benkart-Kang-Kashiwara] ('00), [Kwon] ('12)
- for KM superalgebra with "even" weights [Jeong] ('01)

No examples of canonical bases.

Why believe?

No examples despite extensive study, experts don’t believe.
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Because now we have

- a better definition of U (all h. wt. modules $/\mathbb{Q}(q)$);
- a good bar involution;
- a bar-invariant integral form;
- a *categorical* canonical basis.

This motivates us to try again generalizing Kashiwara.

Crystals

We can define Kashiwara operators $\tilde{e}_i, \tilde{f}_i$.

Let $\mathcal{A} \subset \mathbb{Q}(q)[\pi]$ be the ring of functions with no pole at $q = 0$.

$V(\lambda)$ is said to have a **crystal basis** $(\mathcal{L}, B)$ if

- $\mathcal{L}$ is a $\mathcal{A}$-lattice of $V(\lambda)$ closed under $\tilde{e}_i, \tilde{f}_i$

and $B \subset \mathcal{L}/q\mathcal{L}$ satisfies

- B is a π-basis of $\mathcal{L}/q\mathcal{L}$; (i.e. signed at $\pi = -1$: $B = B \cup \pi B$)
- $\tilde{e}_i B \subseteq B \cup \{0\}$ and $\tilde{f}_i B \subseteq B \cup \{0\}$;
- For $b \in B$, if $\tilde{e}_i b \neq 0$ then $b = \tilde{f}_i \tilde{e}_i b$.

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Canonical Basis

We set

$$V(\lambda) \supset \mathcal{L}(\lambda) = \sum \mathcal{A}\tilde{f}_{i_1} \ldots \tilde{f}_{i_n} v_\lambda, \quad B(\lambda) = \left\{ \pi^{\epsilon} \tilde{f}_{i_1} \ldots \tilde{f}_{i_n} v_\lambda + q\mathcal{L}(\lambda) \right\}$$

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$$(\lambda \in X^+ \cup \{\infty\}, V(\infty) = U^-)$$

Theorem (C-Hill-Wang)

The pairs $(\mathcal{L}(\lambda), B(\lambda))$ for $\lambda \in X^+ \cup \{\infty\}$ are crystal bases. Moreover, there exist maps $G : \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \to \mathcal{L}(\lambda)$ such that $G(B(\lambda))$ is a bar-invariant π-basis of $V(\lambda)$.

We call $G(B(\lambda))$ the canonical basis of $V(\lambda)$.

($\pi = -1$: first canonical bases for quantum supergroups!)

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New idea: a *twistor* (from work with Fan, Li, Wang [CFLW]).

$$U^-|_{\pi=1} \otimes \mathbb{C} \xrightarrow{\cong} U^-|_{\pi=-1} \otimes \mathbb{C}$$

which is *almost* an algebra isomorphism.

Good enough: the ρ-invariance at $\pi = 1$ transports to $\pi = -1$.

Why must the basis be signed?

Example: $I = I_1 = \{i, j\}$ such that $a_{ij} = a_{ji} = 0$.

$$F_i F_j = \pi F_j F_i$$

Should $F_i F_j$ or $F_j F_i$ be in $B(\infty)$? No preferred canonical choice.

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This is not a bad thing!

- A π-basis is an honest $\mathbb{Q}(q)$-basis (for π-free modules)!
- Categorically: represents "spin states" of QHSA modules.

Canonical bases and the whole quantum group

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(Beilinson-Lusztig-McPherson (type A), Lusztig)

$$1 \rightsquigarrow \sum_{\lambda \in X} 1_\lambda \text{ with } 1_\lambda 1_\eta = \delta_{\lambda,\eta} 1_\lambda, \qquad K_i \rightsquigarrow \sum_{\lambda \in X} q^{\langle h_i, \lambda \rangle} 1_\lambda$$

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$\dot{U}$ is the algebra on symbols $x1_\lambda = 1_{\lambda+|x|}x$ for $x \in U$, $\lambda \in X$.

$x1_\lambda$ = projection to λ-wt. space followed by the action of x.

Rank 1

$\dot{U}_q(\mathfrak{osp}(1|2))$ is the algebra given by
Generators: $E1_n = 1_{n+2}E, \quad F1_n = 1_{n-2}F, \quad 1_n$
Relations: $1_n 1_m = \delta_{nm} 1_n, \quad (E1_{n-2})(F1_n) - (F1_{n+2})(E1_n) = [n]1_n$

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Generators: $E1_n = 1_{n+2}E, \quad F1_n = 1_{n-2}F, \quad 1_n$
Relations: $1_n 1_m = \delta_{nm} 1_n, \quad (E1_{n-2})(F1_n) - (F1_{n+2})(E1_n) = [n]1_n$

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We conjectured $\dot{U}_q(\mathfrak{osp}(1|2))$ admits a categorification, and Ellis and Lauda ('13) recently verified our conjecture.

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The canonical basis is stable under the projective limit
$\Rightarrow$ induces a bar-invariant canonical basis on $\dot{U}$.

Further directions

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- Categorification for covering quantum groups
  - Connection to odd link homologies (Khovanov)
  - Tensor modules?
  - Higher rank?

Some Related Papers

Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), pp. 447–498

Kashiwara, *On crystal bases of the Q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), pp. 456–516.

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Ellis, Khovanov, Lauda, *The odd nilHecke algebra and its diagrammatics*, IMRN **2014** pp. 991–1062

Hill and Wang, *Categorication of quantum Kac-Moody superalgebras*, arXiv:1202.2769, to appear in Trans. AMS.

Fan and Li, *Two-parameter quantum algebras, canonical bases and categorifications*, arXiv:1303.2429

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C, Fan, Li, and Wang, *Quantum supergroups III. Twistors*, arXiv:1307.7056, to appear in Comm. Math. Phys.

C, *Quantum supergroups IV. Modified form*, arXiv:1312.4855.

C, Hill, and Wang, *Quantum shuffles and quantum supergroups of basic type*, arXiv:1310.7523.

Slides available at
http://people.virginia.edu/~sic5ag/

Thank you for your attention!