Canonical bases and quantum shuffle superalgebras of basic type

Sean Clark
University of Virginia

(joint with David Hill and Weiqiang Wang)

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QUANTUM GROUPS AND CANONICAL BASES

Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be a simple Lie algebra.

[Lusztig, Kashiwara]: $U_q(\mathfrak{n})$ admits a canonical basis B; that is,

- B is invariant under $q \mapsto \bar{q} = q^{-1}$;
- B equals (any choice of) a PBW basis mod q;
- B is orthonormal (mod q) with respect to a bilinear form.

This basis holds a remarkable amount of information about

- geometry;
- representation theory;
- categorification.

LIE SUPERALGEBRAS

Question: do quantized Lie superalgebras admit canonical bases? (e.g. $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|2n)$, Kac-Moody superalgebras)

No adequate setting for geometric construction à la Lusztig

Theorem (C-Hill-Wang)

The half-quantum supergroup $U_q(\mathfrak{n})$ associated to a Kac-Moody superalgebra with no isotropic roots admits a canonical basis.

Example: $\mathfrak{osp}(1|2n)$ (finite type).
Sketch:

- Set a parameter $\pi^2 = 1$ and a bar involution $\bar{q} = \pi q^{-1}$
- Use π to interpolate between super and non-super
- Use Kashiwara's algebraic (crystal) approach to obtain CB

Basic type

An important class are Lie superalgebras of basic type ($\mathfrak{gl}$, $\mathfrak{osp}$, simple Lie algebras)

- Isotropic simple roots $\Rightarrow$ no root strings
  - In particular, Kashiwara's algebraic strategy fails
- [Benkart-Kang-Kashiwara, Kwon] Many interesting $\mathfrak{gl}(m|n)$-modules admit crystal bases.
- [Khovanov] $\mathfrak{gl}(1|2)$ admits a categorification

Conclusion: We will have to try some new methods

A quantum shuffle approach

Non-super $U_q(\mathfrak{n})$ has been studied using quantum shuffles [Ram-Lalonde, Rosso, Green, Leclerc, ...]

Algebra structure $\rightsquigarrow$ combinatorics of words.

- Order on simple roots induces lexicographic order.
- Certain words (dominant Lyndon) $\leftrightarrow$ positive roots.
- Get distinguished bases from word combinatorics.

Fact: [Leclerc] Canonical bases can be constructed using quantum shuffles

Super canonical bases

Goal: Use quantum shuffles to construct PBW/canonical bases.

For this, we need:

- a quantum shuffle presentation of the quantum group;
- to study super word combinatorics;
- a PBW basis for any order on roots;
- a suitable integral form;

This is a nontrivial generalization:

- Super shuffles lack positivity.
- Super word combinatorics are less well behaved.
- Leclerc's construction is not self-contained (need Lusztig's PBW).

LIE SUPERALGEBRAS

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a simple Lie superalgebra of basic type $(A - G)$.

- $[x, y] = xy - (-1)^{p(x)p(y)} yx$.

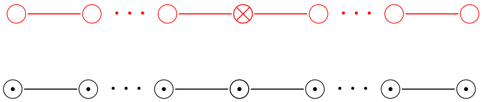
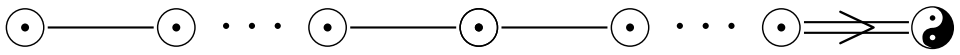
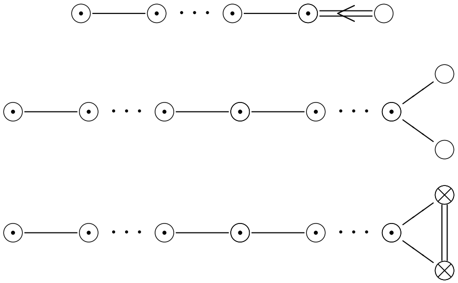
Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$:

- Root system: $\widetilde{\Phi} = \widetilde{\Phi}_{\bar{0}} \sqcup \widetilde{\Phi}_{\bar{1}} = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}$;
  - $\widetilde{\Phi}_{\bar{1}} = \widetilde{\Phi}_{\mathsf{iso}} \sqcup \widetilde{\Phi}_{\mathsf{n-iso}}$;
- Simple roots: $\Pi = \Pi_{\bar{0}} \sqcup \Pi_{\bar{1}} = \{\alpha_i \mid i \in \mathrm{I}\}$,
- Dynkin diagram: (Γ, I),
  - $\mathrm{I} = \mathrm{I}_{\bar{0}} \sqcup \mathrm{I}_{\bar{1}}$, $\mathrm{I}_{\bar{1}} = \mathrm{I}_{\mathsf{iso}} \sqcup \mathrm{I}_{\mathsf{n-iso}}$.

Unlike Lie algebras, (Γ, I) depends on $\mathfrak{h}$, and $\widetilde{\Phi}$ may be unreduced (e.g. type BC).

Reduced root system: $\Phi = \{\alpha \in \widetilde{\Phi} \mid \frac{1}{2}\alpha \notin \widetilde{\Phi}\}$

(Types A-D)

$\mathfrak{gl}(m\|n)$	
$\mathfrak{osp}(2m+1\|2n)$	
$\mathfrak{osp}(2n\|2m)$	

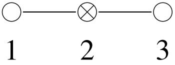
THE HALF-QUANTUM SUPERGROUP

Let $U_q = U_q(\mathfrak{n}^+) = \mathbb{Q}(q)\langle e_i \mid i \in \mathrm{I}\rangle$;

- This is a bialgebra under the multiplication:

$$(u \otimes v)(x \otimes y) = (-1)^{p(v)p(x)} q^{-(|v|,|x|)}(ux \otimes vy),$$

  where $|v|, |x| \in Q^+ = \oplus_{i \in \mathrm{I}} \mathbb{Z}_+ \alpha_i$;
- [Yamane] Subject to (exotic) Serre-type relations determined by subdiagrams of Γ.

Example: 

$$e_1e_2e_3e_2 + e_3e_2e_1e_2 + e_2e_1e_2e_3 + e_2e_3e_2e_1 = (q + q^{-1})e_2e_1e_3e_2$$

BILINEAR FORM

$\mathsf{F} = \mathbb{Q}(q)\langle \mathrm{I} \rangle$, the free algebra on I,

$$\mathbf{i} = (i_1 i_2 \dots i_d) = i_1 \cdot i_2 \cdots i_d, \qquad |\mathbf{i}| = \alpha_{i_1} + \dots + \alpha_{i_d}$$

There is a canonical surjection $\mathsf{F} \longrightarrow U_q$, $i \mapsto e_i$.

Facts: [Lusztig, Yamane]

- $U_q \cong \mathsf{F}/\mathrm{Rad}(\cdot,\cdot)$;
- U_q is equipped with a nondegenerate bilinear form $(\cdot,\cdot)$ satisfying
  1. $(e_i, e_j) = \delta_{ij}$;
  2. $(xy, z) = (x \otimes y, \Delta(z))$;
- the bilinear form on $U_q \otimes U_q$ can be given by

$$(u \otimes v, x \otimes y) = (u, x)(v, y).$$

QUANTUM SHUFFLE EMBEDDING

Coproduct on F induces a product on F^*.

Dualizing $\mathsf{F} \twoheadrightarrow U_q$ induces an injective homomorphism

$$\Psi : U_q \cong U_q^* \hookrightarrow \mathsf{F}^* \cong (\mathsf{F}, \diamond)$$

where $\diamond$ is the **quantum shuffle product**:

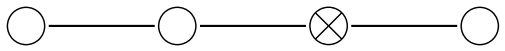
$$(\mathbf{i}\cdot i)\diamond(\mathbf{j}\cdot j) = (\mathbf{i}\diamond(\mathbf{j}\cdot j))\cdot i + (-1)^{(p(\mathbf{i})+p(i))p(j)}q^{-(|\mathbf{i}|+\alpha_i,\alpha_j)}((\mathbf{i}\cdot i)\diamond\mathbf{j})\cdot j.$$

(quantum shuffles approach is dual to Lusztig's bialgebra approach)

We shall study U_q through its image $\mathsf{U} = \Psi(U_q) \subset \mathsf{F}$.

EXAMPLE:

$\mathfrak{gl}(3|2)$



1 2 3 4

$$I_{\bar{0}} = \{1, 2, 4\}, \quad I_{\bar{1}} = I_{\mathrm{iso}} = \{3\}, \quad A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\Psi(e_1 e_2) = 1 \diamond 2 = (21) + q(12)$$

$$\Psi(e_3 e_3) = 3 \diamond 3 = (33) + (-q^0)(33) = 0$$

$$4 \diamond (13) = (134) + q^{-1}(143) + q^{-1}(413)$$

(In fact, $(13) \notin \mathsf{U}$: $\Psi(e_1 e_3) = (13) + (31)$)

TOWARDS PBW BASES

Theorem: [Yamane]
Any basic type Lie superalgebra admits a PBW basis for a ***particular*** *(standard) ordering of the roots.*

But we need PBW bases associated to an *arbitary* ordering of the simple roots.

Idea: Order on simple roots induces a lexicographic order on words.

We can then use the combinatorics of the word basis of F.

Combinatorics of Words

F has a **word basis**

$$\mathsf{W} = \sqcup_{n \geq 0} I^n \subset \mathsf{F},$$

so we can learn things about elements of U by writing them in this basis, e.g.

$$\Psi(e_i e_j) = i \diamond j = (ji) + (-1)^{p(i)p(j)} q^{-(\alpha_i, \alpha_j)}(ij).$$

- Fix the lexicographic order on W relative to *some* ordering $(\mathrm{I}, \leq)$.
- Let W^+ be the set of **dominant words**, i.e words of the form

$$\mathbf{i} = \max(u) \text{ for some } u \in \mathsf{U}.$$

e.g. if $i < j$, then $\max(i \diamond j) = (ji)$.

MONOMIAL BASIS

Proposition:
Let $\mathbf{i} = (i_1, \ldots, i_d) \in \mathsf{W}$ *and* $\varepsilon_{\mathbf{i}} = \Psi(e_{i_1} \ldots e_{i_d})$. *Then*

$$\{\varepsilon_{\mathbf{i}} = i_1 \diamond \cdots \diamond i_d \mid \mathbf{i} = (i_1, \ldots, i_d) \in \mathsf{W}^+\}$$

is a basis of U. We want to refine this basis.

Let L denote the set of **Lyndon words**:

$$\mathbf{i} = (i_1, \ldots, i_d) \in \mathsf{L} \quad \Leftrightarrow \quad \mathbf{i} < (i_k, \ldots, i_d) \text{ for } k > 1.$$

Let $\mathsf{L}^+ = \mathsf{L} \cap \mathsf{W}^+$.

DOMINANT LYNDON WORDS AND POSITIVE ROOTS

Theorem: [C-Hill-Wang]

1. *The map*

$$\mathbf{i} = (i_1, \ldots, i_d) \mapsto \alpha_{i_1} + \cdots + \alpha_{i_d} = |\mathbf{i}|$$

   is a bijection $\mathsf{L}^+ \longrightarrow \Phi^+$;

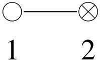
2. *Every* $\mathbf{i} \in \mathsf{W}^+$ *has a canonical factorization* $\mathbf{i} = \mathbf{i}_1 \cdots \mathbf{i}_n$, *where* $\mathbf{i}_1, \ldots, \mathbf{i}_n \in \mathsf{L}^+$, $\mathbf{i}_1 \geq \cdots \geq \mathbf{i}_n$ *and* $\mathbf{i}_k > \mathbf{i}_{k+1}$ *if* $|\mathbf{i}_k| \in \Phi^+_{\mathsf{iso}}$.

Sketch:

- Induct on height of roots
- Yamane's PBW theorem gives dimensions of root spaces
- If no bijection, then $\mathbf{ii} \in \mathsf{W}^+$ with $|\mathbf{i}| \in \Phi_{\mathsf{iso}}$
- This yields a contradiction.

EXAMPLE:

$\mathfrak{gl}(2|1)$



$\mathsf{L}^+ = \{(1), (12), (2)\}$
$\mathsf{W}^+ = \{(2)^a(12)^b(1)^c : c \in \mathbb{N}, a, b \in \{0, 1\}\}$

Words which are dominant:

- (2121)
- (21111)
- (121)
- (1111)

Words which are not dominant:

- (112)
- (22111)
- (1212)
- (1121)

ROOT VECTORS

To construct PBW, we need root vectors $\leftrightsquigarrow$ q-commutators.
Need some prescribed order to inductively build them.

Define the **co-standard factorization** of $\mathbf{i} \in \mathsf{L}^+$ to be $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$, where $\mathbf{i}_1 \in \mathsf{L}^+$ is of maximal length ($\mathbf{i}_2 \in \mathsf{L}^+$, too!).

Example

- $(1234) = (123)(4)$;
- $(12332) = (1233)(2)$;
- $(1231234) = (123)(1234)$.

Definition: For $i \in \mathrm{I}$, set $\mathrm{E}_i = i$. Otherwise, define $\mathrm{E}_{\mathbf{i}}$, $\mathbf{i} \in \mathsf{L}^+$, inductively by

$$\begin{aligned}\mathrm{E}_{\mathbf{i}} &= \kappa_{\mathbf{i}}^{-1}[\mathrm{E}_{\mathbf{i}_2}, \mathrm{E}_{\mathbf{i}_1}]_{q^{-1}} \\ &= \kappa_{\mathbf{i}}^{-1}\left(\mathrm{E}_{\mathbf{i}_2} \diamond \mathrm{E}_{\mathbf{i}_1} - (-1)^{p(\mathbf{i}_1)p(\mathbf{i}_2)} q^{-(|\mathbf{i}_1|, |\mathbf{i}_2|)} \mathrm{E}_{\mathbf{i}_1} \diamond \mathrm{E}_{\mathbf{i}_2}\right)\end{aligned}$$

where

- $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ is the co-standard factorization;
- $\kappa_{\mathbf{i}}$ is an explicit quantum integer defined in terms of Φ^+.

For $\mathbf{i} \in \mathsf{W}^+$ with canonical factorization

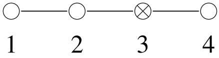
$$\mathbf{i} = \mathbf{i}_1^{k_1} \cdots \mathbf{i}_n^{k_n}, \qquad \mathbf{i}_t \in \mathsf{L}^+ \quad \text{s.t.} \quad \mathbf{i}_1 > \cdots > \mathbf{i}_n,$$

we set

$$\mathrm{E}_{\mathbf{i}} = \mathrm{E}_{\mathbf{i}_n}^{(k_n)} \diamond \cdots \diamond \mathrm{E}_{\mathbf{i}_2}^{(k_2)} \diamond \mathrm{E}_{\mathbf{i}_1}^{(k_1)}.$$

Example

$\mathfrak{gl}(3|2)$



$$\mathsf{L}^+ = \{(i, \dots, j) : 1 \le i \le j \le 4\}$$

$$\begin{aligned} E_{(34)} &= (4) \diamond (3) - q^{-1}(3) \diamond (4) \\ &= (34) + q^{-1}(43) - q^{-1}(43) - q^{-2}(34) = (1 - q^{-2})(34) \end{aligned}$$

$$E_{(23)} = (1 - q^2)(23)$$

$$\begin{aligned} E_{(234)} &= (1 - q^{-2})\left((34) \diamond (2) - q(2) \diamond (34)\right) \\ &= (1 - q^{-2})\big((234) + q(324) + q(342) - q(342) - q(324) - q^2(234)\big) \\ &= -(q - q^{-1})^2(234) \end{aligned}$$

Theorem: [C-Hill-Wang]

1. *The set* $\mathrm{PBW} = \{\mathrm{E}_{\mathbf{i}} \mid \mathbf{i} \in \mathsf{W}^+\}$ *is a basis of* U*:*
   - $\mathrm{E}_{\mathbf{i}} = \varepsilon_{\tau(\mathbf{i})} + \sum_{\mathbf{j}>\mathbf{i}} a_{\mathbf{ij}}\varepsilon_{\tau(\mathbf{j})}$;
2. *for* $\mathbf{i} \in \mathsf{W}^+$, $\max(\mathrm{E}_{\mathbf{i}}) = \mathbf{i}$*;*
3. *for* $\mathbf{i} \in \mathsf{L}^+$,

$$\Delta(\mathrm{E}_{\mathbf{i}}) = \mathrm{E}_{\mathbf{i}} \otimes 1 + \left(\sum_{\substack{|\mathbf{j}|+|\mathbf{k}|=|\mathbf{i}| \\ \mathbf{j}<\mathbf{i}<\mathbf{k}}} \vartheta^{\mathbf{i}}_{\mathbf{j},\mathbf{k}} \, \mathrm{E}_{\mathbf{k}} \otimes \mathrm{E}_{\mathbf{j}} \right) + 1 \otimes \mathrm{E}_{\mathbf{i}};$$

4. PBW *is orthogonal:* $(\mathrm{E}_{\mathbf{i}}, \mathrm{E}_{\mathbf{j}}) = 0$ *if* $\mathbf{i} \neq \mathbf{j}$.

INTEGRAL FORM

If we want to construct a canonical basis, we need a bar-invariant integral form. This integral form should be spanned by the PBW bases.

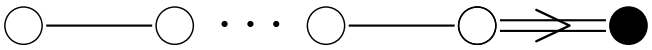
The classical choice is to take the lattice of divided powers.

This is too naive and won't work in general.
(Silly example: $\mathfrak{gl}(n|n)$ with all simples isotropic)

However, it does work for certain standard root systems.

Restricted Simple Systems

From now on, we shall only consider the following systems:

$\mathfrak{gl}(m,n)$	
$\mathfrak{osp}(1,2n)$	
$\mathfrak{osp}(2\|2n)$	

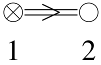
INTEGRAL FORM

Theorem: [C-Hill-Wang]
For these types, PBW $= \{\mathrm{E}_{\mathbf{i}} \mid \mathbf{i} \in \mathsf{W}^{+}\}$ *is a basis for*

$$\mathsf{U}_{\mathcal{A}} = \mathcal{A}\langle e_i^{(n)} \mid i \in \mathrm{I}, n \geq 0\rangle, \quad (\mathcal{A} = \mathbb{Z}[q, q^{-1}]).$$

Remark: Even for standard systems, there are examples where this fails.

$$\mathfrak{osp}(3|2)$$



Pseudo-Canonical Bases

Lemma: [C-Hill-Wang]
For $\mathbf{i} \in \mathsf{W}^+$, *write*

$$\overline{\mathrm{E}_{\mathbf{i}}} = \sum_{\mathbf{j} \in \mathsf{W}^+} a_{\mathbf{ij}} \mathrm{E}_{\mathbf{j}}, \qquad \textit{for } a_{\mathbf{ij}} \in \mathbb{Q}(q).$$

Then, $a_{\mathbf{ii}} = 1$ *for all* $\mathbf{i} \in \mathsf{W}^+$ *and* $a_{\mathbf{ij}} = 0$ *if* $\mathbf{i} > \mathbf{j}$.
(This holds for arbitrary type.)

For our allowed diagrams, $\overline{\mathsf{U}_{\mathcal{A}}} = \mathsf{U}_{\mathcal{A}}$ so all $a_{\mathbf{ij}} \in \mathcal{A}$.
Therefore, by standard argument there exists a unique bar-invariant basis of the form

$$\mathrm{b}_{\mathbf{i}} = \mathrm{E}_{\mathbf{i}} + \sum_{\mathbf{j} > \mathbf{i}} \theta_{\mathbf{ij}} \mathrm{E}_{\mathbf{j}}, \quad \theta_{\mathbf{ij}} \in q\mathbb{Z}[q]$$

Call such a basis a **pseudo-canonical basis**.

CANONICAL BASES

A priori, this pseudo-canonical basis depends on the PBW basis, hence on an ordering.

A pseudo-canonical basis will be called a **canonical basis** if it is **almost orthogonal** in the sense that, for all $\mathbf{i}, \mathbf{j} \in \mathsf{W}^+$,

1. $(\mathrm{b}_{\mathbf{i}}, \mathrm{b}_{\mathbf{j}}) \in \mathbb{Z}[q]$, and
2. $(\mathrm{b}_{\mathbf{i}}, \mathrm{b}_{\mathbf{j}}) = \pm\delta_{\mathbf{ij}} \pmod{q}$ for some $\theta \in \{0, 1\}$.

Theorem:[C-Hill-Wang]
For U *corresponding to the standard system of type* $\mathfrak{gl}(m|n)$*,* $\mathfrak{osp}(1|2n)$*, or* $\mathfrak{osp}(1|2n)$

- U *has a pseudo-canonical basis.*
- *The pseudo-canonical basis is canonical except for* $\mathfrak{gl}(m|n)$ *with both* $m, n > 1$

Remark: Our approach is entirely self contained!

- Though we have focused on super, this works for non-super.
- This is a new self-contained construction of non-super CBs.

What about other types?

Arbitrary root system: is there a good integral form?

- Do PBW lattices coincide?
- Are these lattices bar-invariant?

$\mathfrak{gl}(m|n)$ for $m, n > 1$: We have a pseudo-canonical basis.

- Basis isn't compatible with $U_q(\mathfrak{gl}(n) \oplus \mathfrak{gl}(m))$
- Unlikely to behave well with representations

Crucial case to understand is $\mathfrak{gl}(2|2)$.

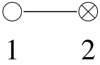
WHAT ABOUT REPRESENTATIONS?

Question: Does the canonical basis descend to a basis on simple modules?

No choice but to compute the image of basis elements.

We will do this for the following system:

$\mathfrak{gl}(2|1)$



CANONICAL BASIS OF $\mathfrak{gl}(2|1)$

Let us work with $U_q(\mathfrak{n}^-)$ instead of $U_q(\mathfrak{n})$.
Using our construction, we compute the canonical basis

$$B = \{F_1^{(r)}, F_1^{(r)}F_2, F_2F_1^{(r+1)}, F_2F_1^{(r+1)}F_2\}$$

(This is equal to Khovanov's categorical CB for $\mathfrak{gl}(2|1)$)

Set

- $\lambda = a\epsilon_1 + b\epsilon_2 + c\epsilon_3$: weight for $\mathfrak{gl}(2|1)$ with $b - a \in \mathbb{Z}_{\geq 0}$.
- $K(\lambda)$: the Kac module (maximal finite-dim. parabolic Verma).
- $L(\lambda)$: the unique simple quotient of $K(\lambda)$.
- v_λ^K, v_λ^L: some highest weight vectors of $K(\lambda)$ and $L(\lambda)$.

How does B act on $K(\lambda)$ and $L(\lambda)$?

Descent of canonical basis

For $b \in M$, let $B(m) = \{bm \neq 0 \mid b \in B\}$.

Proposition:

1. *$B(v_\lambda^K)$ is a basis of $K(\lambda)$.*
2. *$B(v_\lambda^L)$ is a basis of $L(\lambda)$ if and only if $a \neq -1 - c$.*
3. *When $a = -1 - c$, the spans of $F_1^{(r)} F_2 v_\lambda^L$ and $F_2 F_1^{(r)} v_\lambda^L$ coincide and are nonzero for $1 \leq r \leq a - b + 1$.*

In particular, $B(v_\lambda^L)$ is a basis for any "polynomial" weight.

A CONJECTURE

Now consider $\mathfrak{gl}(n|1)$ and extend our earlier notations.

Conjecture:

1. $B(v_\lambda^K)$ is a basis of $K(\lambda)$ for all weights λ.
2. $B(v_\lambda^L)$ is a basis of $L(\lambda)$ for all polynomial weights λ.

SOME QUESTIONS

$K(\lambda)$ and $L(\lambda)$ have crystal bases [Benkart-Kang-Kashiwara, Kwon].
Question: How are the canonical and crystal bases related?

Existence of canonical bases suggests a connection to categorification.
Question: Can $\mathfrak{gl}(n|1)$ or $\mathfrak{osp}(2|2n)$ be categorified?

Thank you for your attention!

Slides available at
http://people.virginia.edu/~sic5ag/