A canonical basis for covering quantum groups

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QUANTUM GROUPS

q: a generic parameter;
$\mathfrak{g}$: a Kac-Moody algebra with simple roots $\Pi = \{\alpha_i : i \in I\}$.
$U_q(\mathfrak{g})$ is the $\mathbb{Q}(q)$ algebra with generators $E_i, F_i, K_i^{\pm 1}$ for $i \in I$.
$U_q(\mathfrak{n}^-)$, the subalgebra generated by F_i.

$U_q(\mathfrak{n}^-)$ has many interesting properties, e.g.

- Lusztig-Kashiwara *canonical basis*;
- categorifications of Khovanov-Lauda and Rouquier;

$U_q(\mathfrak{g})$ admits a categorification for its *modified form* [L, KL, R].

HALF QUANTUM SUPERGROUPS

$\mathfrak{g}$: an anisotropic Kac-Moody superalgebra with $\mathbb{Z}/2\mathbb{Z}$-graded simple roots $\Pi = \Pi_{\bar{0}} \sqcup \Pi_{\bar{1}} = \{\alpha_i : i \in I\}$

$U_q(\mathfrak{n}^-)$: algebra generated by F_i satisfying *super* Serre relations. Was not expected to admit a canonical basis.

Super KLR= quiver Hecke superalgebras
(Ellis-Khovanov-Lauda in rank 1, Kang-Kashiwara-Tsuchioka independently defined the general construction)

[Hill-Wang] $U_q(\mathfrak{n}^-)$ is categorified by QHSA's.
$\Rightarrow$ It has a categorical canonical basis.

Is there an intrinsic canonical basis à la Lusztig, Kashiwara?

Insight from [HW]

Anisotropic super and non-super are formally similar

Key Insight [HW]: use a parameter $\pi^2 = 1$ for super signs

- $\pi = 1 \rightsquigarrow$ non-super case.
- $\pi = -1 \rightsquigarrow$ super case.

There is a bar involution on $\mathbb{Q}(q)^\pi = \mathbb{Q}(q,\pi)/(\pi^2 - 1)$ given by

$$q \mapsto \pi q^{-1} \quad (\pi^2 = 1)$$

and quantum integers

$$[n] = \frac{(\pi q)^n - q^{-n}}{\pi q - q^{-1}}, \quad [n]^!, \quad \begin{bmatrix} n \\ a \end{bmatrix} \in \mathbb{Z}[q, q^{-1}].$$

giving $U_q(\mathfrak{n}^-)$ a suitable bar-invariant integral form.

ANISOTROPIC KM

We consider a KM superalgebra with GCM A indexed by $I = I_0 \coprod I_1$ (simple roots) and satisfying:

- $a_{ij} \in \mathbb{Z}$, $a_{ii} = 2$, $a_{ij} \leq 0$
- there exist positive symmetrizing coefficients d_i ($d_i a_{ij} = d_j a_{ji}$)
- (anisotropy) $a_{ij} \in 2\mathbb{Z}$ for $i \in I_1$

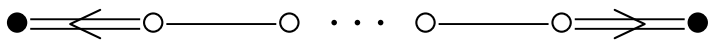
We call these "of anisotropic type". We will also impose:

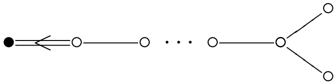
- (bar-compatibility) $d_i \equiv_2 p(i)$

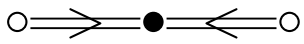
EXAMPLES

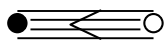
(●=odd root)

 ($\mathfrak{osp}(1|2n)$)









KNOWN FACTS ABOUT KM SUPER

Quantized Lie superalgebras have been well studied (Benkart, Jeong, Kang, Kashiwara, Kwon, Musson, Zou, ...)

Some key coincidences exist for anisotropic KM:

- $\mathfrak{osp}(1|2n)$ reps "=" half of $\mathfrak{so}(2n+1)$ reps (R.B. Zhang, Lanzmann)
- Over $\mathbb{C}(q)$, $U_q(\mathfrak{osp}(1|2n))$ miraculously has the missing reps. (Musson-Zou)

[CW]: $U_q(\mathfrak{osp}(1|2))/\mathbb{Q}(q)$ can be tweaked to get all reps.

$$EF - \pi FE = \underbrace{\frac{K - K^{-1}}{\pi q - q^{-1}}}_{\text{even h.w.}} \quad \text{or} \quad \underbrace{\frac{\pi K - K^{-1}}{\pi q - q^{-1}}}_{\text{odd h.w.}}$$

Definition [CHW1]

Let $\mathfrak{g}$ be a KM superalgebra of anisotropic type, A its symmetrizable GCM.
Let $U = U_q(\mathfrak{g})$ be the $\mathbb{Q}(q)$-algebra with generators E_i, F_i, $K_i^{\pm 1}$, J_i such that

$$J_i^2 = 1, \quad J_i K_i = K_i J_i, \quad J_i J_j = J_j J_i, \quad K_i K_j = K_j K_i,$$

$$J_i E_j J_i^{-1} = \pi^{a_{ij}} E_j, \quad K_i E_j K_i^{-1} = q^{a_{ij}} E_j,$$

$$J_i F_j J_i^{-1} = \pi^{-a_{ij}} F_j, \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j,$$

$$E_i F_j - \pi^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{J_i^{d_i} K_i^{d_i} - K_i^{-d_i}}{(\pi q)^{d_i} - q^{-d_i}};$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \pi^{p(k;i,j)} E_i^{(1-a_{ij}-k)} E_j E_i^{(k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k \pi^{p(k;i,j)} F_i^{(1-a_{ij}-k)} F_j F_i^{(k)} = 0,$$

where $p(k;i,j) = kp(i)p(j) + \frac{1}{2}k(k-1)p(i)$.

Rank 1

For $U_q(\mathfrak{osp}(1|2))$

Generators: $E, F, K^{\pm 1}, J$

Relations:

$$J^2 = 1, \quad JK = KJ,$$

$$JEJ^{-1} = E, \quad KEK^{-1} = q^2 E,$$

$$JFJ^{-1} = F, \quad KFK^{-1} = q^{-2} F,$$

$$EF - \pi F_j E_i = \frac{JK - K^{-1}}{\pi q - q^{-1}};$$

(If h is the Cartan element, $K = q^h$ and $J = \pi^h$.)

We call this covering quantum $\mathfrak{osp}(1|2)$ or $\mathfrak{sl}(2)$ ($\pi = 1$ case)

FINITE TYPE

The only finite type covering algebras have Dynkin diagrams

This diagram corresponds to

- the Lie superalgebra $\mathfrak{osp}(1|2n)$
- the Lie algebra $\mathfrak{so}(1+2n)$

(NB. There is no "covering $\mathfrak{sl}(n)$" in this construction)

STRUCTURES IN A COVERING QUANTUM GROUP

U has all the nice features you could hope for:

- $U = U^- \otimes U^0 \otimes U^+$;
- U^- admits a nondegenerate bilinear form;
- there is a Hopf superalgebra structure (super sign $\mapsto \pi$);
- there is a bar involution ($K \mapsto JK^{-1}$);
- there is a quasi-R-matrix and Casimir-type operator;

REPRESENTATIONS

Let P (P^+) be the set of (dominant) weights of $\mathfrak{g}$.

A weight module is a $U_q(\mathfrak{g})$-module $M = \bigoplus_{\lambda \in P} M^\lambda$, where

$$M^\lambda = \left\{ m \in M : K_i m = q^{\langle h_i, \lambda \rangle} m, \quad J_i m = \pi^{\langle h_i, \lambda \rangle} m \right\}.$$

We can define highest-weight and integrable modules as usual to obtain a semi-simple category $\mathcal{O}_{\text{int}}$.

Simple modules: $V(\lambda)$ for all $\lambda \in P^+$
(Same character as in classical case)

Crystals

To construct a CB, we use the algebraic approach with crystals. Specifically, we construct a covering analogue for

- Kashiwara operators $\tilde{e}_i, \tilde{f}_i$;
- the crystal lattice;
- the action of the q-Boson algebra;
- the polarizations (= deformed Shapovalov forms);
- the tensor product rule;

Kashiwara's grand loop argument can be extended to the covering case.

Moreover, this crystal basis admits globalization to a canonical basis.

Canonical Basis

Theorem (C-Hill-Wang)

U^- and the integrable modules admit compatible canonical bases.

Let B be the canonical basis of U^-.

- If v_λ is the highest weight vector of $V(\lambda)$,

$$bv_\lambda = 0 \text{ or is a CB element.}$$

- $B|_{\pi=1} =$ the Lusztig-Kashiwara CB
- B is typically π-signed: $b \in B$ implies $\pi b \in B$.

Example: $a_{ij} = 0, p(i) = p(j) = 1$

$$F_iF_j = \pi F_jF_i$$

(Categorically: M is not isomorphic to its parity shift ΠM.)

MODIFIED FORM

Basic idea: $1 \rightsquigarrow \sum_{\lambda \in P} 1_\lambda$ with $1_\lambda 1_\eta = \delta_{\lambda,\eta} 1_\lambda$

For $x \in U$, let $|x|$ be the weight.

$\dot{U}$ is the algebra on symbols $x1_\lambda = 1_{\lambda+|x|}x$ for $x \in U$, $\lambda \in P$ satisfying

$$(xy)1_\lambda = x1_{\lambda+|y|}y1_\lambda, \quad J_\mu K_\nu 1_\lambda = \pi^{\langle \mu,\lambda \rangle} q^{\langle \nu,\lambda \rangle} 1_\lambda$$

Any weight U-module M is a $\dot{U}$ module:
$x1_\lambda$ acts as projection to M^λ followed by the U-action of x.

SOME PROPERTIES

$\dot{U}$ has some additional useful properties:

- Automorphisms of U extend to $\dot{U}$;
- $\dot{U}1_\lambda \overset{v.s.}{=} U^- \otimes U^+$;

Theorem (C.)

There is a non-degenerate symmetric bilinear form on $\dot{U}$ which:

- *extends the form on U^-;*
- *is invariant under our favorite maps;*
- *is a limit of polarizations;*

For $\pi = 1$, this is Lusztig's form on $\dot{U}$.

Rank 1

$\dot{U}_q(\mathfrak{osp}(1|2))$ is the algebra given by
Generators: $E1_n = 1_{n+2}E, \quad F1_n = 1_{n-2}F, 1_n$
Relations: $1_n1_m = \delta_{nm}1_n$ and $EF1_n - FE1_n = [n]1_n$

Theorem (C-Wang)
$\dot{U}_q(\mathfrak{osp}(1|2))$ *admits a canonical basis*

$$\dot{B} = \left\{ E^{(a)}1_nF^{(b)}, \pi^{ab}F^{(b)}1_nE^{(a)} \mid a+b \geq n \right\}.$$

(In rank 1, the basis need not be π-signed)

Ellis and Lauda have categorified $\dot{U}_q(\mathfrak{osp}(1|2))$.

Constructing the CB

- $\dot{U}1_{\lambda-\lambda'}$ projects "nicely" onto $N(\lambda, \lambda')$ (highest weight $\otimes$ lowest weight);
- $N(\lambda, \lambda')$ has a bar involution (Lusztig quasi-$\mathcal{R}$-matrix);
- $N(\lambda, \lambda')$ admits a CB (bar involution + CB on simples);
- The CB of $N(\lambda, \lambda')$ is compatible with $N(\lambda + \lambda'', \lambda'' + \lambda')$;

These facts allow us to build a basis for $\dot{U}$.

Canonical Basis

Theorem (C)

$\dot{U}$ admits a π-signed canonical basis generalizing the basis for U^-.
This basis is π-almost orthonormal under the bilinear form.
For $\pi = 1$, this specializes to Lusztig's canonical basis for $\dot{U}|_{\pi=1}$.

Some Related Papers

[HW] *Categorication of quantum Kac-Moody superalgebras*, arXiv:1202.2769, to appear in Trans. AMS.

[CW] *Canonical basis for quantum* $\mathfrak{osp}(1|2)$, arXiv:1204.3940, Lett. Math. Phys. 103 (2013), 207–231.

[CHW1] *Quantum supergroups I. Foundations*, arXiv:1301.1665, to appear in Trans. Groups.

[CHW2] *Quantum supergroups II. Canonical Basis*, arXiv:1304.7837.

[C] *Quantum supergroups IV. Modified form*, forthcoming.

Slides available at

http://people.virginia.edu/~sic5ag/

Thank you for your attention!